Name: MATH 3 FINAL

Student ID: ANSWER KEY

MATH 3: Final Exam

Problem 1. (6 points) Determine whether the following statements are **TRUE** or **FALSE**. No justification is required.

(a) (1 point) The equation $x^2 + y^2 = 1$ defines y as a function of x.

Answer: FALSE

(b) (1 point) A one-to-one function always has an inverse.

Answer: TRUE

(c) (1 point) The lines $y = 2x + 1$ and $y = -2x - 1$ are parallel.

Answer: FALSE

(d) (1 point) For every real number x, $\ln(e^x) = x$.

Answer: TRUE

(e) (1 point) For every real number x, $\sin^{-1}(\sin(x)) = x$.

Answer: FALSE

(f) (1 point) The function $f(x) = \tan^{-1}(x)$ is one-to-one.

Answer: TRUE

Problem 2. (12 points) Let $f(x) = 2x + 3$.

(a) (3 points) Calculate the x-intercept of $f(x)$.

$0 = f(x) = 2x + 3 \implies \boxed{x = -\frac{3}{2}}$

(b) (3 points) Find the equation of the line $g(x)$ that is perpendicular to $f(x)$ and passes through the point $(2, -3)$.

Let $g(x) = mx + b$. Then $m = -\frac{1}{2}$ and $-3 = g(2) = (-\frac{1}{2}) \cdot 2 + b$

$\implies b = -2$.

$$g(x) = -\frac{1}{2}x - 2$$

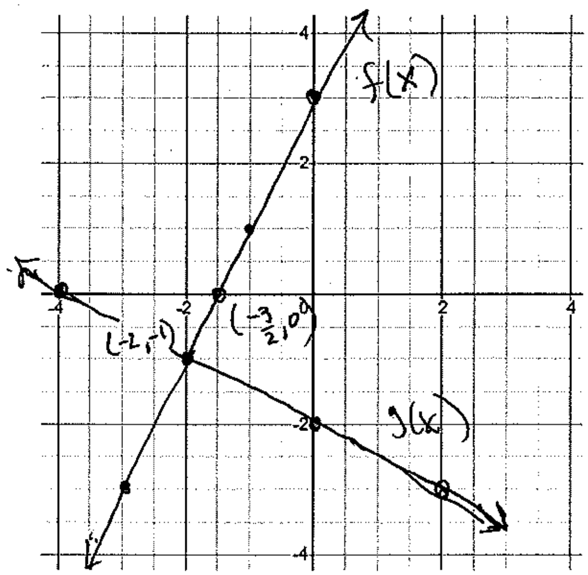
(c) (3 points) Calculate the point where $f(x)$ and $g(x)$ intersect.

Solve $f(x) = g(x)$

$2x + 3 = -\frac{1}{2}x - 2 \implies \frac{5x}{2} = -5 \implies x = -2$

Since $f(-2) = -1$, the lines intersect at $(-2, -1)$

(d) (3 points) Graph both lines on the grid below:



Problem 3. (13 points) Consider the quadratic function $f(x) = 2x^2 + 2x - 4$ given in general form.

(a) (4 points) Identify the vertex, the line of symmetry, and the x and y-intercepts of $f(x)$.

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \boxed{\left(-\frac{1}{2}, -\frac{9}{2}\right)}$

L.O.S: $\boxed{x = -\frac{1}{2}}$

x-int: $f(x) = 2x^2+2x-4 = (2x-2)(x+2) = 0 \iff \boxed{x = 1, -2}$

y-int: $\boxed{f(0) = -4}$

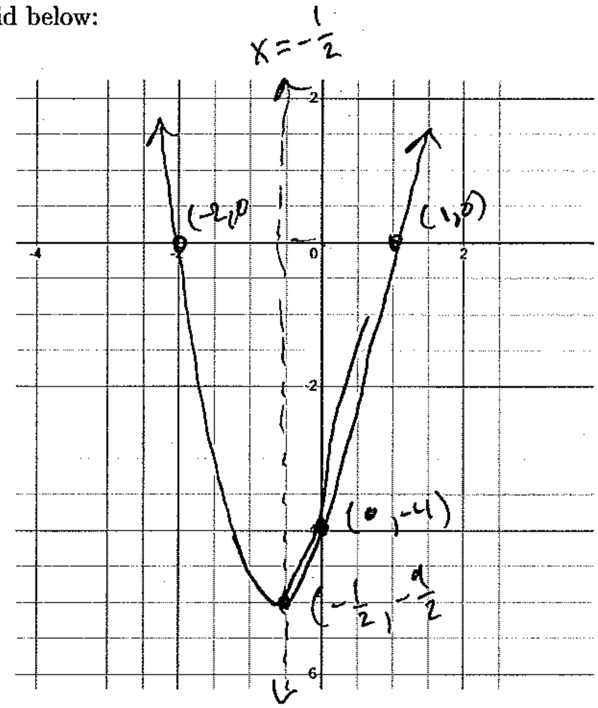
(b) (3 points) Identify the range of $f(x)$.

Since $a = 2 > 0$, parabola opens up $\Rightarrow$ vertex is a local minimum $\Rightarrow$ $\boxed{\text{Range } f(x) = [-9/2, \infty)}$

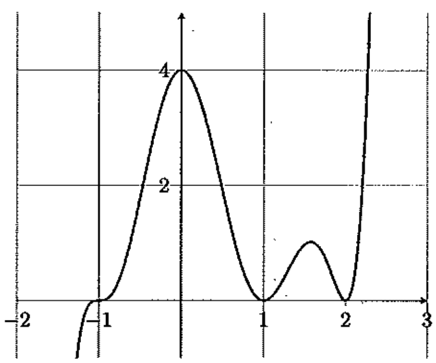
(c) (3 points) Write $f(x)$ in standard form.

$f(x) = a(x-h)^2 + k = \boxed{2\left(x + \frac{1}{2}\right)^2 - \frac{9}{2}}$

(d) (3 points) Graph $f(x)$ on the grid below:



Problem 4. (10 points) Consider the following graph of a polynomial.



Identify the degree, the end-behaviour, the zeros and their multiplicities, and the y-intercept. Write down an equation of smallest degree for this polynomial.

End-Behavior: $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -\infty$ as $x \to -\infty$

$\Rightarrow$ degree is odd

Zeros: $x=-1$, odd degree ≥ 3

$x=1$, even degree ≥ 2 $\qquad \Rightarrow$ degree ≥ 7

$x=2$, even degree ≥ 2

y-int: $f(0)=4$. Now we know $f(x) = a(x+1)^3(x-1)^2(x-2)^2$.

$4 = f(0) = a(0+1)^3(0-1)^2(0-2)^2 \Rightarrow a = 1$.

So $\boxed{f(x) = (x+1)^3(x-1)^2(x-2)^2}$

Problem 5. (8 points) Let $f(x) = x^6 + x^3 + x^2 + 1$.

(a) (6 points) Divide $f(x)$ by $x^3 + 1$ using long division.

$$\begin{array}{r|l} & x^3 \\ \hline x^3+1 & x^6 + x^3 + x^2 + 1 \\ & -(x^6 + x^3) \downarrow \quad \downarrow \\ \hline & 0 \quad \boxed{x^2+1} \end{array}$$

$\leftarrow$ degree is smaller than 3

So $f(x) = (x^3+1)\cdot x^3 + x^2 + 1$

(b) (2 points) Identify the quotient $q(x)$ and the remainder $r(x)$.

$q(x) = x^3 \qquad r(x) = x^2+1$

Problem 6. (9 points) Let $f(x) = 2x^3 - 3x^2 + 1$.

(a) (3 points) List all possible rational zeros of $f(x)$.

$\frac{p}{q}$ where p divides 1 and q divides 2. $p = \pm 1$ $q = \pm 1, \pm 2$

$\frac{p}{q} = \pm 1, \pm \frac{1}{2}$

(b) (3 points) Determine all rational zeros of $f(x)$.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & 0 & 1 \\ & \downarrow & 2 & -1 & -1 \\ \hline & 2 & -1 & -1 & \boxed{0} \end{array} \quad \boxed{f(1)=0}$$

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 0 & 1 \\ & \downarrow & -2 & 5 & -5 \\ \hline & 2 & -5 & 5 & \boxed{-4} \end{array} \quad f(0)=-4$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -3 & 0 & 1 \\ & \downarrow & 1 & -1 & -\frac{1}{2} \\ \hline & 2 & -2 & -1 & \boxed{1/2} \end{array} \quad f(1/2) = \frac{1}{2}$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & 0 & 1 \\ & \downarrow & -1 & 2 & -1 \\ \hline & 2 & -4 & 2 & \boxed{0} \end{array}$$

$\boxed{f(-1/2) = 0}$

$\boxed{x = 1 \text{ and } x = -1/2 \text{ are the only rational zeros}}$

(c) (3 points) Factor $f(x)$ as a product of three linear (degree 1) polynomials.

Using $f(1)=0$ computation from (b):

$$f(x) = (x-1)\underbrace{(2x^2-x-1)}_{g(x)}.$$

Using $f(-1/2)=0$, we know $g(-1/2)=0$. Divide:

$$\begin{array}{r|rrr} -1/2 & 2 & -1 & -1 \\ & \downarrow & -1 & 1 \\ \hline & 2 & -2 & 0 \end{array}$$

$\Rightarrow g(x) = (x+1/2)(2x-2) = (2x+1)(x-1)$

$\Rightarrow \boxed{f(x) = (x-1)^2(2x+1)}$

Problem 7. (8 points) Let $f(x) = \frac{(x-1)^2(x+2)(x+3)}{(x+2)(x-3)}$.

(a) (2 points) Determine the x and y intercept(s) of $f(x)$.

x-int: $0 = f(x) \Leftrightarrow 0 = (x-1)^2(x+2)(x+3)$

$\Leftrightarrow x=1,\ x=-2,\ x=-3.$

But $f(-2)$ is undefined, so this is not an x-int.

$\Rightarrow \boxed{x=1, x=-3}$

y-int: $f(0) = \frac{(-1)^2(2)(3)}{2 \cdot -3} = -1$

(b) (2 points) Determine the vertical asymptote(s) of $f(x)$.

$x=3$

Note that $x=-2$ is NOT a vertical asymptote because $(x+2)$ is a factor of numerator and denominator.

(c) (2 points) Determine the horizontal asymptote of $f(x)$. If there is no horizontal asymptote, explain why.

degree of numerator $\geq$ degree of denominator $+2$.

$\Rightarrow$ $\boxed{\text{No horizontal asymptote}}$

(d) (2 points) Determine the removable discontinuity of $f(x)$, if one exists.

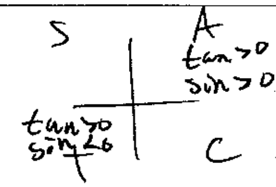
$X = -2$

Problem 8. (12 points) Suppose that $\tan\theta = \frac{12}{5}$ and $\sin(\theta) < 0$.

(a) (2 point) Determine which quadrant θ lies in.

$\tan\theta > 0$ and $\sin\theta < 0$

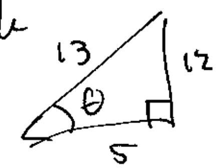
$\Rightarrow$ $\boxed{\theta \text{ is in } Q\,\mathrm{III}}$



(b) (2 points) Find $\sec\theta$.

$\boxed{\sec\theta = -\frac{13}{5}}$

use this triangle in QIII $\rightarrow$



(c) (2 points) Find $\cot\theta$.

$\boxed{\cot\theta = \frac{5}{12}}$

(d) (2 points) Find $\csc\theta$.

$\boxed{\csc\theta = -\frac{13}{12}}$

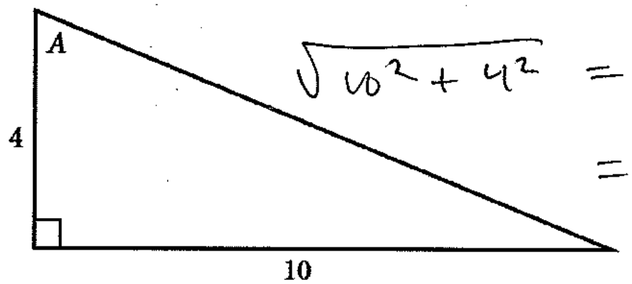
(e) (2 points) Find $\sin\theta$.

$\boxed{\sin\theta = -\frac{12}{13}}$

(f) (2 points) Find $\cos\theta$.

$\boxed{\cos\theta = -\frac{5}{13}}$

Problem 9. (14 points) Consider the following right triangle.



$$\sqrt{10^2+4^2} = \sqrt{116}$$
$$= 2\sqrt{29}$$

(a) (2 points) Find the length of the hypotenuse using the Pythagorean Theorem.

$2\sqrt{29}$ from above

(b) (2 points) Find $\sin(A)$.

$$\sin A = \frac{10}{2\sqrt{29}} = \frac{10\sqrt{29}}{2 \cdot 29} = \frac{5\sqrt{29}}{29}$$

(c) (2 points) Find $\cos(A)$.

$$\cos A = \frac{4}{2\sqrt{29}} = \frac{2\sqrt{29}}{29}$$

(d) (2 points) find $\tan(A)$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{10}{4} = \frac{5}{2}$$

(e) (2 points) find $\cot(A)$.

$$\cot A = \frac{2}{5}$$

(f) (2 points)Find $\sec(A)$.

$$\sec A = \frac{\sqrt{29}}{2}$$

(g) (2 points) Find $\csc(A)$.

$$\csc A = \frac{\sqrt{29}}{5}$$

Problem 10. (12 points) Let $f(x) = \pi\cos(\pi x - \pi)$ and $g(x) = \pi\sec(\pi x - \pi)$.

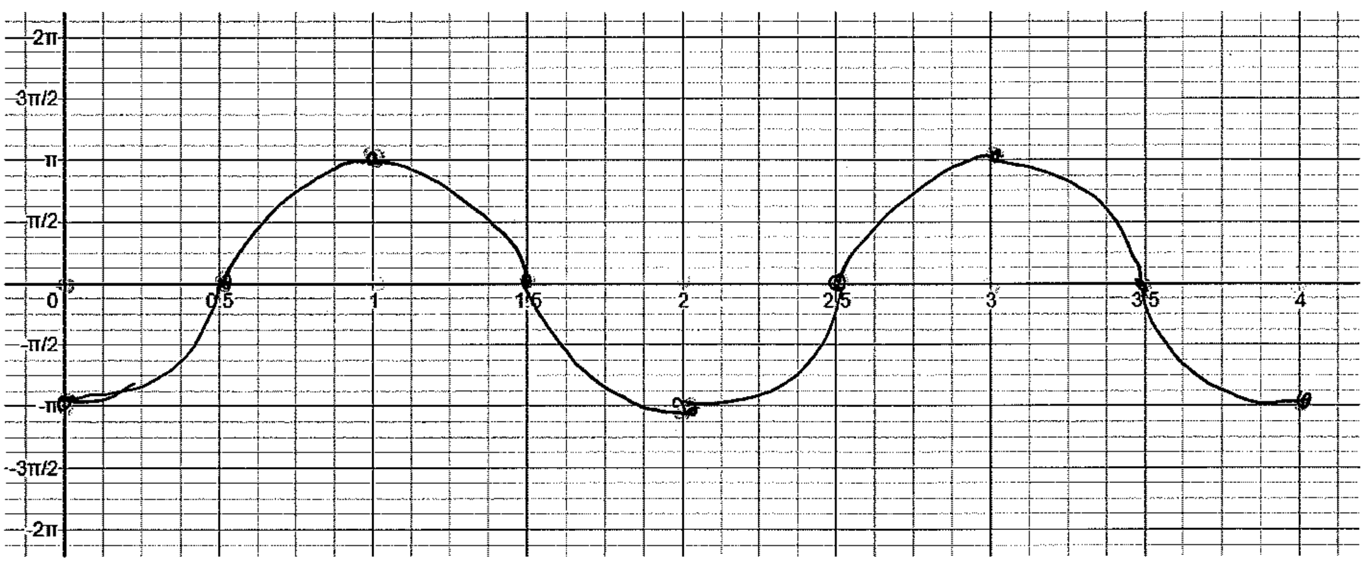
$= \pi \cos(\pi(x-1))$

(a) (4 points) Identify the amplitude, period, midline, and phase shift of the sinusoidal function $f(x)$.

amp $= \pi$ Period $= \frac{2\pi}{\pi} = 2$ midline $y = 0$

phase shift: right 1 unit

(b) (3 points) Graph two full periods of $f(x)$ on the interval $[0, 4]$ below

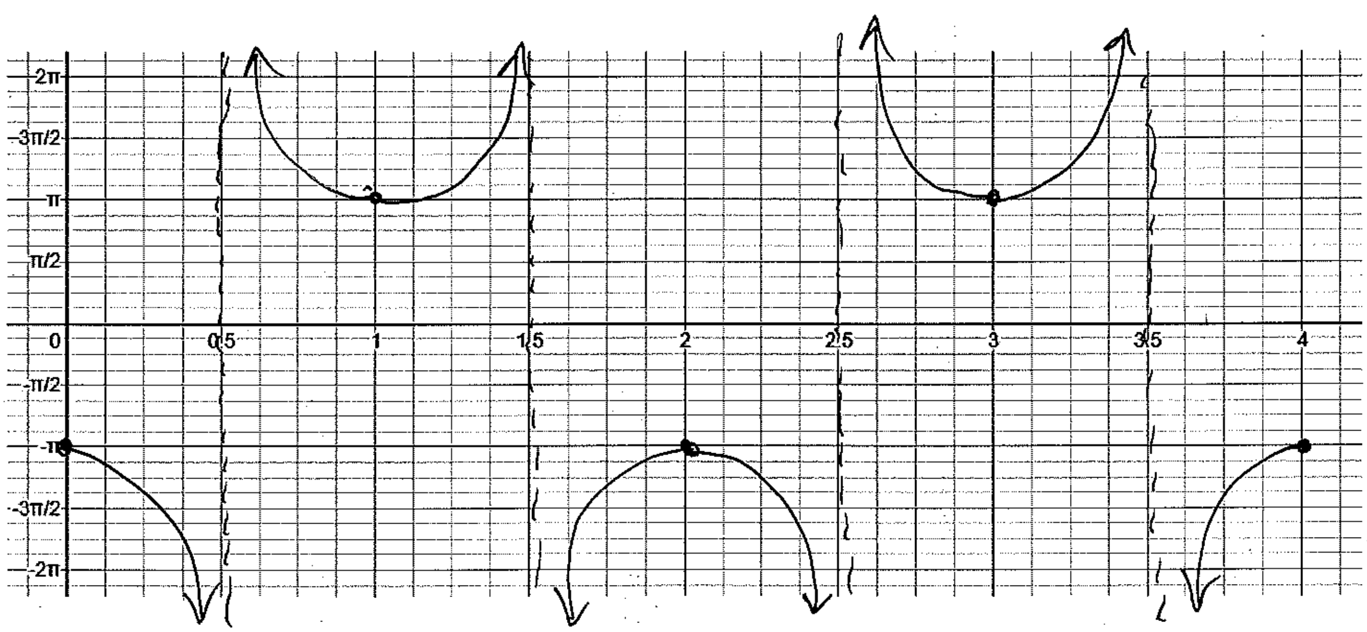


(c) (2 points)Identify *all* vertical asymptotes of the periodic function $g(x)$.

g(x) has an asymptote whenever $f(x) = 0$

$\Rightarrow$ $x = \frac{1}{2} + k$ k is any integer

(d) (3 points) Graph two full periods of $g(x)$ on the interval $[0, 4]$ below. Use vertical dotted lines to identify the vertical asymptotes. Hint: the graph can be obtained from the graph in part (b).



Problem 11. (10 points) Find the exact value of the following expressions.

(a) (2 points) $\sin^{-1}(\cos(\pi))$

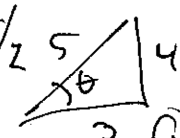
$$\sin^{-1}(\cos\pi) = \sin^{-1}(-1) = -\frac{\pi}{2}$$

(b) (2 points) $\tan^{-1}(\sin([illegible]))$ ← $\frac{3\pi}{2}$

$$\tan^{-1}(\sin 3\pi/2) = \cancel{[illegible]}$$
$$= \tan^{-1}(-1) = -\pi/4$$

(c) (2 points) $\cos(\sin^{-1}(-4/5))$

Set $\theta = \sin^{-1}\left(-\frac{4}{5}\right) \Rightarrow \sin\theta = \frac{-4}{5}$, $-\pi/2 \le \theta \le \pi/2$



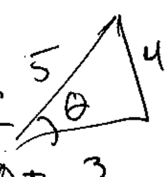
Q IV

$$\boxed{\cos(\sin^{-1}(-4/5)) = \cos\theta = \frac{3}{5}}$$

$\cos\theta > 0$ in Q III

(d) (2 points) $\sin(\tan^{-1}(4/3))$

Set $\theta = \tan^{-1}(4/3) \Rightarrow \tan\theta = 4/3$, $-\frac{\pi}{2} \le \theta \le \pi/2$



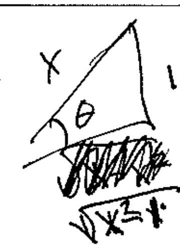
QI

$$\boxed{\sin(\tan^{-1}(4/3)) = \sin\theta = +\frac{4}{5}}$$

$\sin\theta > 0$ in QI

(e) (2 points) $\cos(\sin^{-1}(1/x))$ Note: your final answer for this one should be a function of x.

Set $\theta = \sin^{-1}(1/x) \Rightarrow \sin\theta = \frac{1}{x}$ $-\pi/2 \le \theta \le \pi/2$



QI or QIV

$$\boxed{\cos(\sin^{-1}(1/x)) = \cos\theta = \frac{\sqrt{x^2-1}}{x}}$$

Problem 12. (8 points) Find the exact value of the following expressions. Hint: write each angle as a sum or difference of two *special* angles and then use a sum or difference identity.

(a) (4 points) $\sin(11\pi/12)$

$$\sin(11\pi/12) = \sin\left(\frac{8\pi}{12} + \frac{3\pi}{12}\right) =$$

$$= \sin\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sin\frac{2\pi}{3} \cdot \cos\frac{\pi}{4} + \cos\frac{2\pi}{3} \cdot \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}(\sqrt{3}-1)}{4}}$$

(b) (4 points) $\cos(11\pi/6)$

$$\cos(11\pi/6) = \cos\left(\frac{5\pi}{6} + \frac{6\pi}{6}\right)$$

$$= \cos\left(\frac{5\pi}{6} + \pi\right)$$

$$= \cos 5\pi/6 \cos\pi - \sin 5\pi/6 \sin\pi$$

$$= -\frac{\sqrt{3}}{2} \cdot -1 - \frac{1}{2} \cdot 0 = \boxed{\frac{\sqrt{3}}{2}}$$

Problem 13. (12 points) Find all solutions θ for the following trigonometric equations. Note: there may be infinitely many solutions.

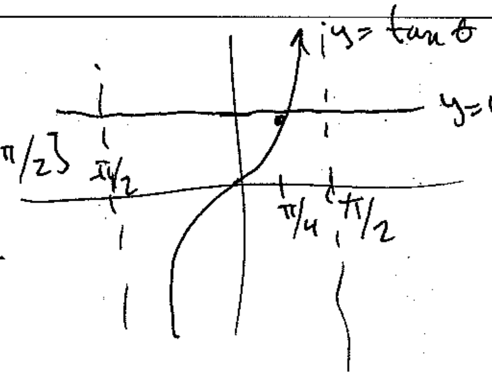
(a) (6 points) $\cot\theta = 1$.

$\cot\theta = 1 \iff \tan\theta = 1$

$\theta = \pi/4$ is a solution on $\{-\pi/2, \pi/2\}$

The period of $\tan\theta$ is π.

So all solutions are:



$\boxed{\theta = \pi/4 + k\pi, \ k \text{ is any integer}}$

(b) (6 points) $\sin(2\theta) = \cos(\theta)$. Hint: use an identity to simplify.

Use $\sin 2\theta = 2\sin\theta\cos\theta$. We have

$$2\sin\theta\cos\theta = \sin(2\theta) = \cos\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

From unit circle, $\theta = \pi/6$, $11\pi/6$ on $[0, 2\pi]$.
The period of sine is 2π, so all solutions are:

$$\boxed{\theta = \pi/6 + 2k\pi \quad \theta = 11\pi/6 + 2k\pi}$$